Lecture 6

6.6 - Inverse Trigonometric Functions

$\arcsin x$ $(= \sin^{-1} x)$

Restricting the domain of $\sin x$ to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ we can make it one-to-one. The range is still $[-1, 1]$. This allows us to define an inverse sine function: $\arcsin(x)$

$\mathcal{D}(\arcsin x) = [-1, 1]$ $\qquad$ $\mathcal{R}(\arcsin x) = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Ex: Evaluate $\tan(\underbrace{\arcsin(\tfrac{\sqrt{3}}{2})}_{\theta})$

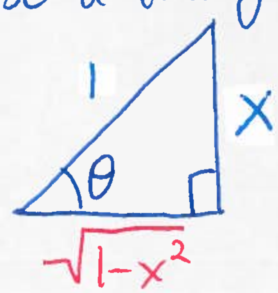
$\sin \theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{3}$

$\Rightarrow \tan(\arcsin(\frac{\sqrt{3}}{2})) = \tan(\frac{\pi}{3}) = \sqrt{3}$

Ex: Find a formula in terms of x for $\cos(\underbrace{\arcsin(x)}_{\theta})$

Use a triangle: $\quad \sin \theta = \frac{\text{opp.}}{\text{hyp.}} = x$

$\cos \theta = \frac{\text{adj.}}{\text{hyp.}}$



$\cos(\arcsin x) = \cos(\theta) = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$

If $\arcsin x = y \Rightarrow x = \sin y$

Differentiating gives: $1 = (\cos y)\, y'$

$$\Rightarrow y' = \frac{1}{\cos y} = \frac{1}{\cos(\arcsin x)}$$

So, $$\boxed{\frac{d}{dx}\left(\arcsin x\right) = \frac{1}{\sqrt{1-x^2}}}$$

and the chain rule gives

$$\boxed{\frac{d}{dx}\left(\arcsin(g(x))\right) = \frac{g'(x)}{\sqrt{1-[g(x)]^2}}}$$

which holds for x values in the domain of g and such that $-1 \le g(x) \le 1$.

Ex: Differentiate $f(x) = \arcsin(e^x)$. What is the domain of $f(x)$?

Sol By the chain rule $\boxed{f'(x) = \frac{e^x}{\sqrt{1-e^{2x}}}}$

Domain: Want $-1 \le e^x \le 1$. e^x is always >0, and $e^x = 1$ when $x=0$, so $\boxed{D(f) = (-\infty, 0]}$

$\arccos x\ (= \cos^{-1} x)$

To make $\cos x$ one-to-one, we restrict the domain to $[0, \pi]$. This lets us define $\arccos x$.

$\mathcal{D}(\arccos x) = [-1, 1]$ $\quad \mathcal{R}(\arccos x) = [0, \pi]$

Similarly to $\arcsin x$, we can find the derivative:

$$\boxed{\frac{d}{dx}(\arccos(x)) = \frac{-1}{\sqrt{1-x^2}}}$$

$\arctan x\ (= \tan^{-1} x)$

Restricting to one period of $\tan x$: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we can make it one-to-one, with a range of: $(-\infty, \infty)$

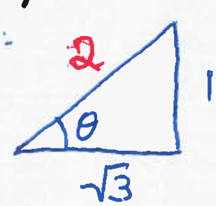
Thus, the inverse tangent function: $\arctan x$ satisfies: $\mathcal{D}(\arctan x) = (-\infty, \infty)$ $\quad \mathcal{R}(\arctan x) = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Ex: Evaluate $\sin\left(\arctan\left(\frac{1}{\sqrt{3}}\right)\right)$.

Sol: Can use triangles for these too:

$\theta = \arctan\left(\frac{1}{\sqrt{3}}\right)$

$\tan\theta = \frac{\text{opp.}}{\text{adj.}} = \frac{1}{\sqrt{3}}$

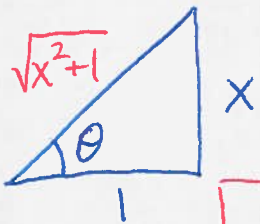


$$\boxed{\sin\left(\arctan\left(\frac{1}{\sqrt{3}}\right)\right) = \frac{1}{2}}$$

Writing $y = \arctan x$, we have $\tan y = x$.

Differentiating: $(\sec^2 y)y' = 1$

$$\Rightarrow y' = \frac{1}{\sec^2 y} = \cos^2 y = \cos^2(\arctan x)$$



$$= \left(\frac{1}{\sqrt{x^2+1}}\right)^2 = \frac{1}{x^2+1}$$

So, $\boxed{\frac{d}{dx}(\arctan x) = \frac{1}{x^2+1}}$

Ex: Let $f(x) = \arctan(\ln(2-x^2))$. What is the domain of $f(x)$? What is $f'(x)$?

Sol:

Need $-\infty < \ln(2-x^2) < \infty$, which it always is.

Need $0 < 2-x^2 \Rightarrow x^2 < 2 \Rightarrow -\sqrt{2} < x < \sqrt{2}$

$\boxed{D(f) = (-\sqrt{2}, \sqrt{2})}$

$$f'(x) = \frac{1}{(\ln(2-x^2))^2+1} \cdot (\ln(2-x^2))'$$

$$= \frac{1}{(\ln(2-x^2))^2+1} \cdot \frac{1}{2-x^2} \cdot (-2x)$$

$$= \boxed{\frac{-2x}{(2-x^2)[(\ln(2-x^2))^2+1]}}$$

Integration

We can reverse the derivatives we found above to get:

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$$

$$\int \frac{1}{1+x^2}\,dx = \arctan x + C$$

Ex: Compute

$$\int \frac{1}{x^2+a^2}\,dx = \int \frac{1}{a^2}\left(\frac{1}{\left(\frac{x}{a}\right)^2+1}\right)dx \overset{\left(u=\frac{x}{a}\right)}{=\!=} \int \frac{1}{a^2}\left(\frac{1}{u^2+1}\right)(a\,du)$$

$$= \frac{1}{a}\int \frac{1}{1+u^2}\,du = \frac{1}{a}\arctan u + C$$

$$= \boxed{\frac{1}{a}\arctan\left(\frac{x}{a}\right) + C}$$

Ex: Compute

$$\int \frac{e^{2x}}{\sqrt{1-e^{4x}}}\,dx \overset{\left(u=e^{2x}\right)}{=\!=} \int \frac{1/2}{\sqrt{1-u^2}}\,du = \frac{1}{2}\arcsin u + C$$

$$= \boxed{\frac{1}{2}\arcsin\left(e^{2x}\right) + C}$$